Late-Time Evolution of Charged Gravitational Collapse and Decay of Charged Scalar Hair - III. Nonlinear Analysis

Shahar Hod and Tsvi Piran

The Racah Institute for Physics, The Hebrew University, Jerusalem 91904, Israel

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Abstract

We study the *nonlinear* gravitational collapse of a *charged* massless scalar-field. We confirm the existence of oscillatory inverse power-law tails along future timelike infinity, future null infinity and along the future outer-horizon. The *nonlinear* dumping exponents are in excellent agreement with the *analytically* predicted ones. Our results prove the analytic conjecture according to which a *charged* hair decays *slower* than a neutral one and also suggest the occurrence of mass-inflation along the Cauchy horizon of a *dynamically* formed charged black-hole.

I. INTRODUCTION

Linearized *perturbation* analysis have revealed important dynamical features of the gravitational collapse of massless neutral fields. In particular, according to the linearized perturbation theory there are two major features which characterize the *late* stages of the evolution: quasinormal ringing and inverse power-law tails (which follows the quasinormal ringing). These late-time tails are relevant to two major aspects of black-hole physics: The *no-hair theorem* of Wheeler and the *mass-inflation* scenario [1]. The mechanism responsible for the development of these neutral inverse power-law tails was first studied by Price [2]. This work was further extended by Gundlach, Price and Pullin [3]. These authors have

also shown that the nonlinear dumping exponents, describing the fall-off of a massless neutral scalar-field at late-times, are in good agreement with the prediction of the linearized perturbation analysis [4].

The late-time evolution of a *charged* massless scalar-field and the physical mechanism for the radiation of a *charged* hair were first studied *analytically* in [5,6], hereafter referred to as papers I and II, respectively. The main result presented there is the existence of oscillatory inverse power-law tails along the asymptotic regions of future timelike infinity, future null infinity and along the future outer-horizon with *smaller* (compared to the neutral case) dumping exponents. These dumping exponents depend on the black-hole's (or the star's) charge. In this paper we study numerically the fully *nonlinear* gravitational collapse of a massless *charged* scalar-field. We confirm that the late-time behaviour of the fully *nonlinear* evolution is in excellent agreement with the *analytical* predictions of the linearized analysis.

The plan of the paper is as follows. In Sec. II we give a short review of the linearized *analytical* results of papers I and II, on which we base our expectations for the fully nonlinear case. In Sec. III we describe our physical system, namely the coupled Einstein-Maxwell-charged scalar equations. In Sec. IV we study the late-time evolution of the *nonlinear* spherical charged gravitational collapse. In Sec. V we study the late-time evolution of non-spherical perturbations on a fixed Reissner-Nordström background and on a time-dependent background. In all cases we compare the *nonlinear* results with the predictions of the linearized *analytical* theory. We conclude in Sec. VI with a brief summary of our results and their physical implications.

II. REVIEW OF LINEARIZED ANALYTICAL RESULTS

We study numerically the late-time behaviour of the fully *nonlinear* gravitational collapse of a massless *charged* scalar-field. Our expectations are based on the *linearized analytical* results of papers I and II. Due to the *smallness* of the field's amplitude at late times, we expect these results to hold even in the fully *nonlinear* case. In other wards, we expect

that the late-time field may be regarded as a small perturbation. Thus, our expectations include the existence of inverse *power-law* tails along the three asymptotic regions: timelike infinity i_+, future null infinity $scri_+$ and along the black-hole outer-horizon H_+ (where the power-law is multiplied by a *periodic* term). Quantitatively, in paper II it was found that the *late-time* behaviour of linearized *charged*-scalar perturbations (on a Reissner-Nordström background) is dominated by power-law tails of the form

$$\psi_m^l = A_{ti}(l, eQ) y^{\beta+1} t^{-(2\beta+2)} \ , \tag{1}$$

at timelike infinity i_+,

$$\psi_m^l = A_{ni}(l, eQ) v_e^{-ieQ} u_e^{-(\beta+1-ieQ)} \ , \tag{2}$$

at future null infinity $scri_+$ and

$$\psi_m^l = A_{eh}(l, eQ) e^{i\frac{eQ}{r_+}y} v_e^{-(2\beta+2)} \ , \tag{3}$$

along the black-hole outer-horizon H_+, where the charged scalar-field ϕ is given by

$$\phi = e^{ie\Phi t} \sum_{l,m} \psi_m^l \left(t, r\right) Y_l^m(\theta, \varphi)/r \ , \tag{4}$$

where Φ is merely a gauge constant of the electromagnetic potential A_t, i.e. its value at infinity (A_t is given by $A_t = \Phi - \frac{Q}{r}$). The constant $\beta(l, eQ)$ is given by

$$\beta = \frac{-1 + \sqrt{(2l+1)^2 - 4(eQ)^2}}{2} \ . \tag{5}$$

The tortoise radial coordinate y is defined by $dy = dr/\lambda^2$, where $\lambda^2 = 1 - \frac{2M}{r} + \frac{Q^2}{r^2}$. Here $v_e \equiv t+y$ and $u_e \equiv t-y$ are the ingoing and outgoing Eddington-Finkelstein null coordinates, respectively. The expressions for the coefficients $A(l, eQ)$ are given in paper II.

III. THE EINSTEIN-MAXWELL-CHARGED SCALAR EQUATIONS

We consider a spherically-symmetric self-gravitating *charged* scalar-field ϕ. The system is described by the coupled Einstein-Maxwell-charged scalar equations. This physical system

was already described in a previous paper [7]. Here we give only a short description of the system and the final form of the equations studied. We express the metric of a spherically symmetric spacetime in the form [8,9]

$$ds^2 = -g(u,r)\bar{g}(u,r)du^2 - 2g(u,r)dudr + r^2 d\Omega^2 \ , \tag{6}$$

in which u is a retarded time null coordinate and the radial coordinate r is a geometric quantity which directly measures surface area. The coordinates have been normalized so that u is the proper time on the $r = 0$ central world line. We use the auxiliary field $\tilde{\phi}$ defined by

$$\phi \equiv \frac{1}{r}\int_0^r \tilde{\phi} \ dr \ , \tag{7}$$

in terms of which the Einstein equations yield

$$g(u,r) = exp\left[4\pi\int_0^r \frac{(\tilde{\phi}-\phi)(\tilde{\phi}^* - \phi^*)}{r}dr\right] \ , \tag{8}$$

and

$$\bar{g}(u,r) = \frac{1}{r}\int_0^r \left(1 - \frac{Q^2}{r^2}\right) g \ dr \ . \tag{9}$$

The electromagnetic potential A_u is given by the Maxwell equations

$$A_u = \int_0^r \frac{Q}{r^2} g \ dr \ , \tag{10}$$

where the charge $Q(u,r)$ within a sphere of radius r, at a retarded time u is given by

$$Q(u,r) = 4\pi ie\int_0^r r(\phi^*\tilde{\phi} - \phi\tilde{\phi}^*)dr \ . \tag{11}$$

The mass $M(u,r)$ within a sphere of radius r, at a retarded time u is given by

$$M(u,r) = \int_0^r \left[2\pi\frac{\bar{g}}{g}(\tilde{\phi}-\phi)(\tilde{\phi}^* - \phi^*) + \frac{1}{2}\frac{Q^2}{r^2}\right] dr + \frac{1}{2}\frac{Q^2}{r} \ . \tag{12}$$

Finally, the wave-equation for the charged scalar-field takes the form of a pair of coupled differential equations

$$\frac{d\tilde{\phi}}{du} = \frac{1}{2r}(g-\bar{g})(\tilde{\phi}-\phi) - \frac{Q^2}{2r^3}(\tilde{\phi}-\phi)g - \frac{ieQ}{2r}g\phi - ie\tilde{\phi}A_u \ , \tag{13}$$

and

$$\frac{dr}{du} = -\frac{1}{2}\bar{g} \ . \tag{14}$$

We solve this system of equations numerically. A detailed description of our algorithm, numerical methods, discretization and error analysis are given in Ref. [7].

In order to compare our *nonlinear* results with the linearized *analytical* results of papers I and II, we use another spacetime coordinate, namely the Bondi time $t_B \equiv t(u,\infty)$, which is the retarded time coordinate that agrees with time at infinity. The proper time $t(u,r)$ along an $r = const$ trajectory is given by [4]

$$t(u,r) \equiv \int_0^u \sqrt{g(u',r)\bar{g}(u',r)}du' \ . \tag{15}$$

The comparison with the analytical results also requires the usage of the ingoing and outgoing Eddington-Finkelstein null coordinates v_e and u_e. In the asymptotic region $r \gg M_{BH}$, v_e is linear with r (along a $u = const$ ray). In particular, $v_e \simeq 2y \simeq 2r$ along the $u = u_e = 0$ ray. In a similar way u_e is linear with t along a $v = const$ ray, namely $u_e = 2t - v_e$ (one may also use the asymptotic relation $u_e \simeq v_e - 2r$ along a $v = const$ ray in order to evaluate u_e). The relation between A_u and A_t implies that $A_t(r_+) = 0$, i.e. $\Phi = \frac{Q}{r_+}$.

IV. NONLINEAR SPHERICAL COLLAPSE

In this section we present our results for the *nonlinear* spherically symmetric gravitational collapse of the self-gravitating massless *charged* scalar-field. We have focused our attention on the behaviour of the charged field ϕ along the three asymptotic regions: timelike infinity i_+, null infinity $scri_+$ and the black-hole outer horizon H_+. The late-time evolution of a charged scalar-field is independent of the form of the initial data. The numerical results presented here have an initial profile of the form

$$\phi(u=0,r) = Ar^2 exp\left\{-\left[(r-r_0)/\sigma\right]^2\right\} \ , \tag{16}$$

where $r_0 = 2.5, \sigma = 0.5$ for the real part of the complex field and $r_0 = 3.0, \sigma = 0.5$ for the imaginary part. Figure 1 displays the time evolution of ϕ_R, the real part of the charged scalar-field, along these asymptotic regions for a *supercritical* evolution ($A = 0.005, e = 0.85$) in which a black-hole forms. The mass and charge of the formed black-hole are $M_{BH} = 0.503$ and $Q_{BH} = -0.420$, respectively. The top panel shows the behaviour of the field at a constant radius (here $r = 10$) as a function of the Bondi time. The middle panel shows the behaviour of the field along null infinity (approximated by the null surface $v = v_{max}$, where v_{max} is the largest value of v on the grid) as a function of u_e. The bottom panel shows the behaviour of the field along the black-hole outer horizon H_+ (approximated by the null surface $u = u_{max}$, where u_{max} is the largest value of u on the grid) as a function of v_e. Initially, the evolution is dominated by the prompt contribution and by the quasinormal ringing. However, at late-times a definite oscillatory *power-law* fall off is manifest. The *nonlinear* power-law exponents (determined from the maximas of these oscillations) are -1.78 at timelike infinity, -0.86 along null infinity and -1.96 along the black-hole outer horizon. These values should be compared with the *analytically* predicted values of $-1.70, -0.85, -1.70$, [see Eqs. $(1) - (3)$], respectively. The theoretical values of the oscillation frequencies of the charged scalar field ϕ are $\frac{eQ_{BH}}{r_+}$ at timelike infinity and along the black-hole outer horizon and $\frac{eQ_{BH}}{2r_+}$ along null infinity [see Eq. (4) and the relation $\Phi = \frac{Q}{r_+}$]. These values agree to within 2% with the numerical estimates.

While the dumping exponents and the decay-rate of neutral perturbations are independent of the spacetime parameters (M and Q), the analysis of paper II predicts that the *charged* dumping exponents depend on the black-hole's (or the star's) *charge* (namely, on the dimensionless quantity $|eQ|$). In order to test this linearized *analytical* prediction, we have studied the dependence of the charged dumping exponents at timelike infinity on the parameter e. The initial form of the field is given by (16) and the amplitude is set to $A = 0.005$. The nonlinear results are summarized in table I. We find a good agreement

between the numerically measured exponents and the linearized predicted ones.

The analysis of paper I predicts the existence of charged power-law tails even in subcritical evolution when there is no collapse to a black-hole (and all we have are imploding and exploding shells). This phenomena is related to the fact that the late-time evolution of the field is dominated by the backscattering from asymptotically *far* regions, and it does *not* depend on the small-r details of the spacetime (this is also the situation for neutral massless perturbations [3]). Since the charge of the spacetime is a *dynamical* quantity it is not clear which value of Q should be taken in calculating the value of the dumping exponent. However, since the charge of the spacetime falls to zero asymptotically we expect the effective value of $|eQ|$ to be much *smaller* than unity. In this case the dumping exponents are expected to be $2l+2$ at timelike infinity and $l+1$ along null infinity (with small corrections of order $O[(eQ)^2]$). Figure 2 displays the time evolution of ϕ_R for a *subcritical* evolution (A=0.002, e=3). Shown are the behaviour of the field at a constant radius (here $r = 10$) as a function of Bondi time and along null infinity as a function of u_e. The *nonlinear* power-law exponents are -2.00 at timelike infinity and -1.00 along null infinity. These values are exactly equal to those predicted in paper I.

V. NON-SPHERICAL PERTURBATIONS OF SPHERICAL COLLAPSE

In order to study the dependence of the late-time dumping-exponents on the multipole index l we have performed two series of numerical investigations. First, we have integrated the *linearized charged* scalar-field equation

$$\psi_{,tt} + 2ie\frac{Q}{r}\psi_{,t} - \psi_{,yy} + V\psi = 0 \ , \tag{17}$$

where

$$V = V_{M,Q,l,e}\left(r\right) = \left(1 - \frac{2M}{r} + \frac{Q^2}{r^2}\right)\left[\frac{l\left(l+1\right)}{r^2} + \frac{2M}{r^3} - \frac{2Q^2}{r^4}\right] - e^2\frac{Q^2}{r^2} \ , \tag{18}$$

on a *fixed* Reissner-Nordström background. It is straightforward to integrate Eq. (17) using the method described in [3]. The late-time evolution of a charged scalar-field is independent

of the form of the initial data. The results presented here are of a Gaussian pulse on $u=0$

$$\psi(u=0,v)=Aexp\left\{-\left[(v-v_0)/\sigma\right]^2\right\}\ , \tag{19}$$

where the amplitude A is physically irrelevant due to the linearity of Eq. (17). It should be noted that the evolution equation (17) is invariant under the rescaling

$$r\to ar\ ,\ t\to at\ ,\ M\to aM\ ,\ Q\to aQ\ ,\ e\to e/a\ , \tag{20}$$

where a is some positive constant. The black-hole mass and charge are set equal to $M_{BH}=0.5$ and $Q_{BH}=0.45$, respectively. We have chosen $e=0.01$, $v_0=100$ and $\sigma=20$ (for both the real and the imaginary parts of the charged field). The numerical results for the l=0,1 and 2 modes are shown in Fig. 3. (from top to bottom, respectively). This figure demonstrates the dependence of the late-time tails at i_+ (here y=400) on the multipole index l. A definite *power-law* fall off is manifest at late-times. The numerical values of the power-law exponents, describing the fall-off of the field at *late* times are $-1.97, -3.94$ and -5.75 for l=0,1 and 2, respectively. These values are to be compared with the *analytically* predicted values of $-2.0, -4.0$ and -6.0, respectively.

In a second series of numerical investigation, we have studied the evolution of a second perturbative charged scalar-field ξ evolved on a *time-dependent* spacetime. This time-dependent spacetime is determined by the background solution ϕ, while we ignore the contribution of the field ξ to the energy-momentum tensor (this is analogous to the case studied in [4] for a neutral field). Resolving the field into spherical harmonics $\xi=\sum_{l,m}\xi^l_m(t,r)Y^m_l(\theta,\varphi)$ (and using the spherical symmetry of the time-dependent background) one obtains a wave-equation for each multiple moment of the field ξ

$$\frac{d\tilde{\xi}^l_m}{du}=\frac{1}{2r}(g-\bar{g})(\tilde{\xi}^l_m-\xi^l_m)-\frac{Q^2}{2r^3}(\tilde{\xi}^l_m-\xi^l_m)g-\frac{ieQ}{2r}g\xi^l_m-ie\tilde{\xi}^l_mA_u-\frac{1}{2r}l(l+1)g\xi^l_m\ , \tag{21}$$

[together with Eq. (14)], where

$$\xi^l_m\equiv\frac{1}{r}\int_0^r\tilde{\xi}^l_m dr\ . \tag{22}$$

The quantities $g, \bar{g}$ and Q are still determined by the background solution for the field ϕ. For the background field ϕ we choose the same initial form as in Sec. IV. The most interesting time-dependent backgrounds are the *non*-collapsing ones. Thus, we take the *subcritical* initial conditions A=0.002 and e=3 of Sec. IV. For the perturbation field ξ we choose initial data of the form Eq. (16) with $r_0 = 4.0$ and $\sigma = 0.5$ for both the real and the imaginary parts of the field (again, the amplitude of the perturbation field is physically irrelevant). Figure 4 displays the time evolution of the perturbative charged scalar field ξ (its real part) at a constant radius (here $r = 10$) as a function of Bondi time. It is interesting that even for *time-dependent* spacetimes the late-time behaviour of charged-fields is well described by an inverse *power-law* fall-off. The numerical values of the power-law exponents are -1.99 for the l=0 mode (top panel) and -4.16 for the l=1 mode (bottom panel). These values should be compared with the fixed background *analytically* predicted values of -2.0 and -4.0, respectively.

VI. SUMMARY AND PHYSICAL IMPLICATIONS

We have studied the *nonlinear* gravitational collapse of a massless *charged* scalar-field. Following the predictions of the linearized *analytical* theory (papers I and II) we have focused attention on the *asymptotic* late-time evolution of the charged-field. Our main results and their physical implications are:

We have confirmed the existence of oscillatory inverse *power-law tails* in a *collapsing* spacetime along the asymptotic regions of future timelike infinity i_+, future null infinity $scri_+$ and along the future outer-horizon H_+. The *nonlinear* dumping exponents are in excellent agreement with the *analytically* predicted ones. In particular, we have verified the analytically conjectured dependence of the *charged* dumping exponents on the dimensionless quantity $|eQ|$. This dependence on the spacetime charge is contrasted to neutral perturbations, where the dumping exponents are fixed integers which does *not* depend on the spacetime parameters (namely, they are functions of the multipole index l only). We have

confirmed the existence of power-law tails even in non-collapsing spacetimes (i.e. imploding and exploding shells).

We have also studied the late-time evolution of non-spherical perturbations (*charged* test fields) on a fixed Reissner-Nordström background and on a time-dependent background. In both cases we have found that the dumping exponents, describing the fall-off of the charged field at late-times, are in good agreement with the predictions of the linearized analytical theory (In particular, we have verified the functional dependence of the dumping exponents on the multipole index l).

Our nonlinear results verify the analytic conjecture that a *charged* hair decays *slower* than a neutral one. Furthermore, the existence of oscillatory inverse *power-law* tails along the black-hole outer horizon suggests the occurrence of the well-known phenomena of *mass-inflation* [1] along the Cauchy horizon of a *dynamically* formed charged black-hole.

In a forthcoming paper we study the *fully nonlinear* gravitational collapse of a charged scalar-field, using a different numerical scheme which is based on *double* null coordinates. This scheme allows us to start with *regular* initial conditions (at approximately past null infinity), calculating the *formation* of the regular black-hole's event horizon, and continue the evolution all the way *inside* the black-hole. Thus, this numerical scheme makes it possible to test the mass-inflation conjecture during the gravitational collapse of a *charged* scalar-field. To our knowledge, the mass-inflation scenario has *never* been demonstrated explicitly before in *collapsing* situations (The numerical work of Brady and Smith [10] begins on a Reissner-Nordström spacetime and the black-hole formation itself was *not* calculated there).

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FIGURES

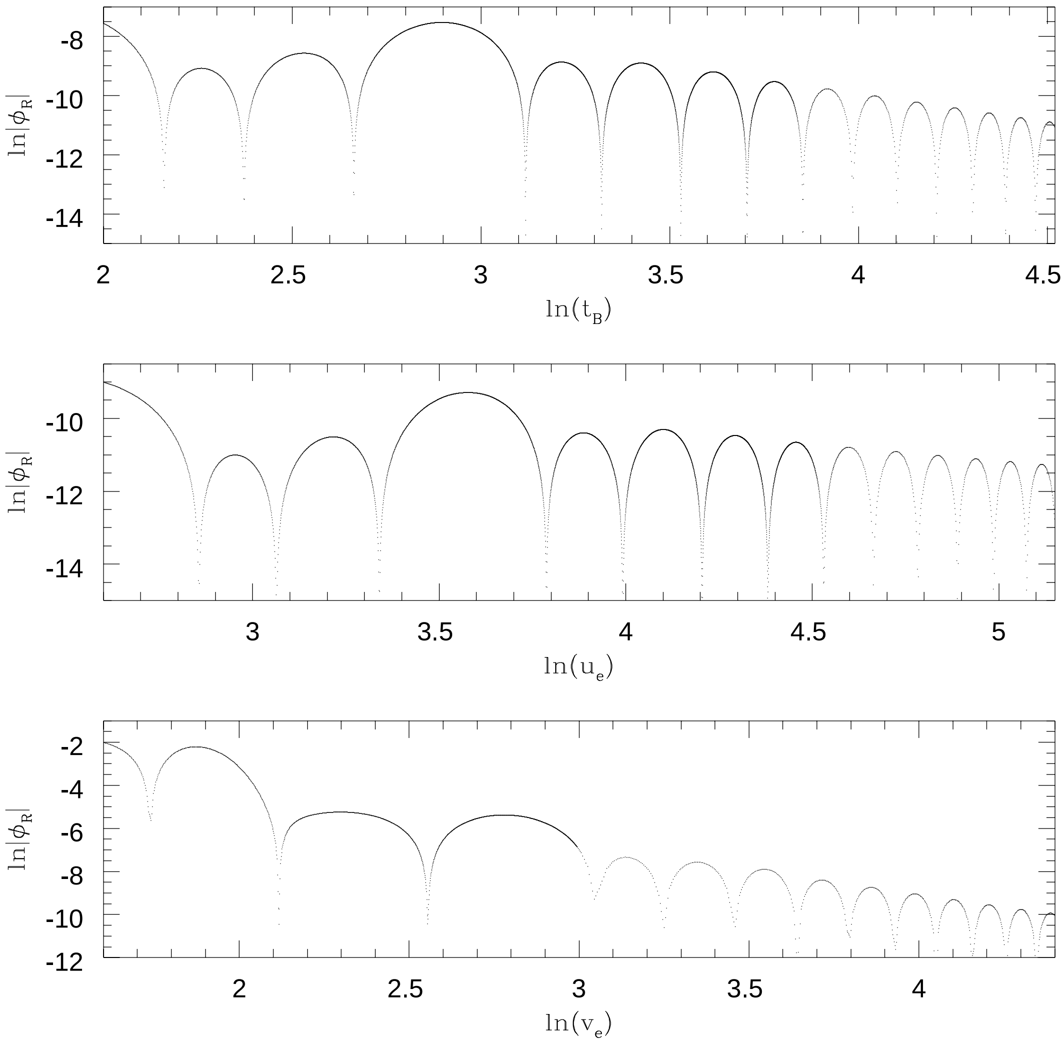

FIG. 1. Supercritical evolution of the charged field $|\phi_R|$ along the asymptotic regions of timelike infinity (top panel), null infinity (middle panel) and the black-hole outer horizon (bottom panel). The initial data is a Gaussian distribution (multiplied by an r^2 factor). The field's amplitude is A=0.005 and e=0.85. The mass and charge of the formed black-hole are $M_{BH} = 0.503$ and $Q_{BH} = -0.420$, respectively. A definite oscillatory *power-law* fall-off is manifest at late-times. The *nonlinear* power-law exponents are -1.78 at timelike infinity (here $r = 10$), -0.86 along null infinity and -1.96 along the black-hole outer horizon. These values are to be compared with the *analytically* predicted values of $-1.70, -0.85$ and -1.70, respectively. The oscillation frequency of the charged field ϕ agrees with the *analytically* predicted value to within 2%.

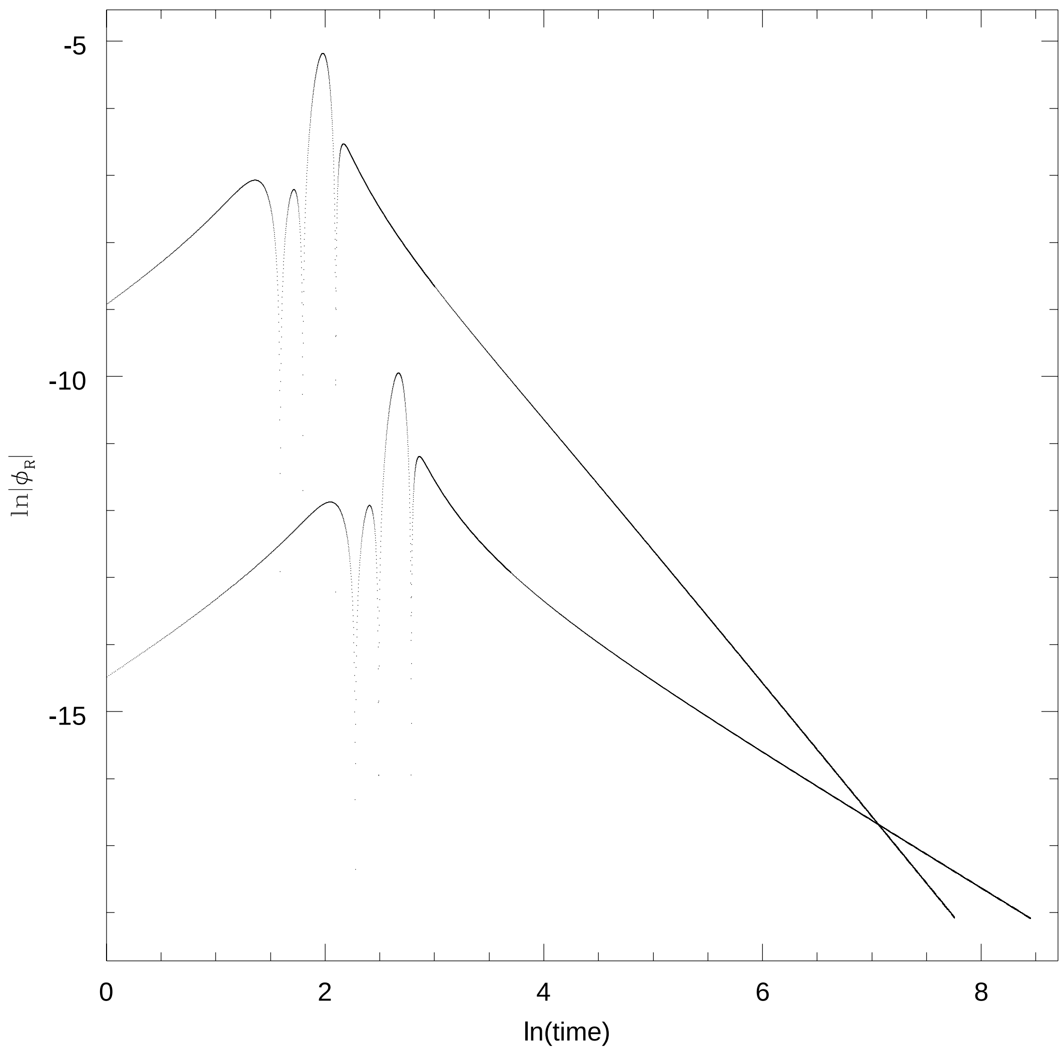

FIG. 2. Subcritical evolution of the charged field $|\phi_R|$. The initial form of the field is the same as in Fig. 1. The field's amplitude is A=0.002 and e=3. The field at future timelike infinity ($r = 10$) is shown as a function of t_B. Along null infinity the field is shown as a function of u_e. A definite *power-law* fall-off is manifest at late-times. The *nonlinear* power-law exponents are -2.00 at timelike infinity and -1.00 along null infinity. These values are exactly the ones predicted by the linearized *analytical* approach.

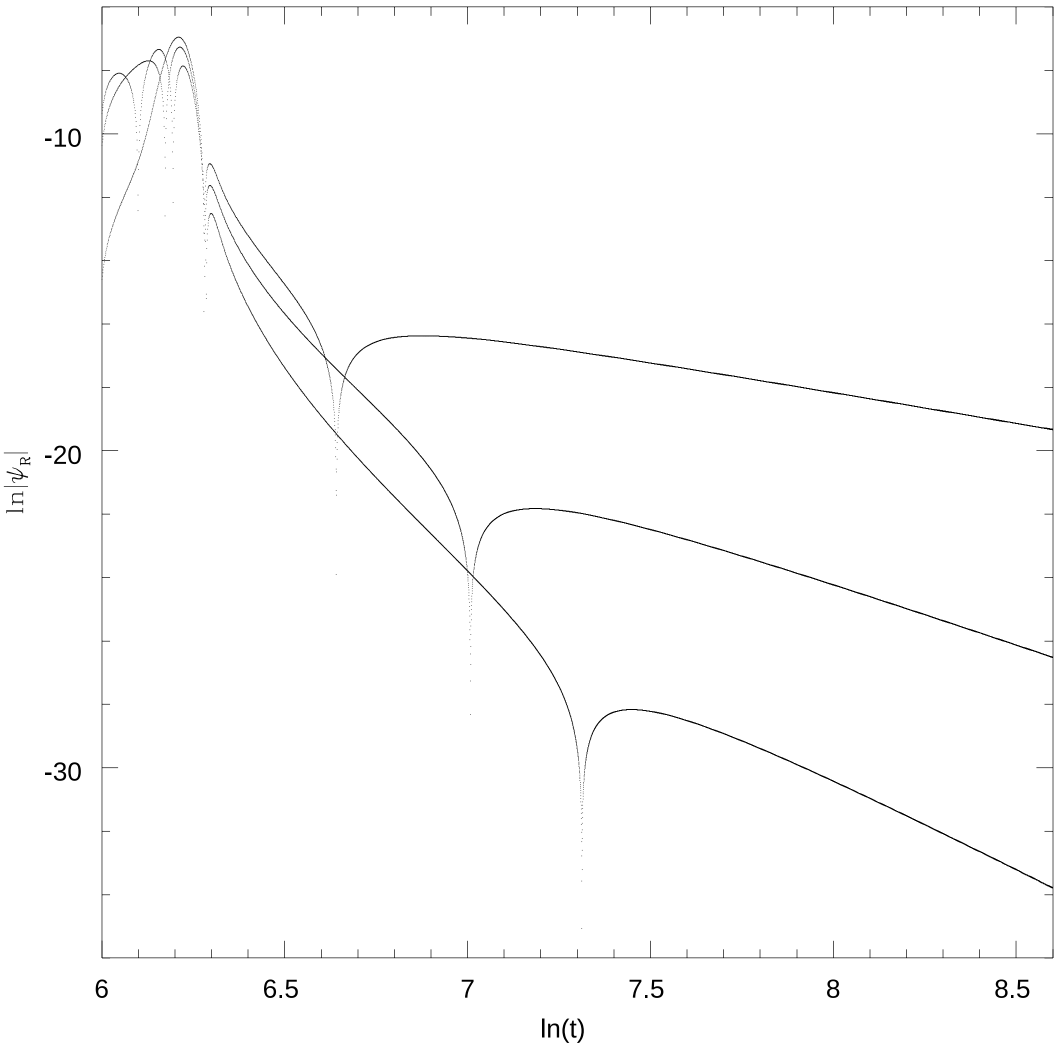

FIG. 3. Evolution of the charged field $|\psi_R(y = 400, t)|$ on a fixed Reissner-Nordström background for different multipoles l=0,1 and 2 (from top to bottom, respectively). The black-hole mass and charge are set equal to $M_{BH} = 0.5$ and $Q_{BH} = 0.45$, respectively. The field's charge is $e = 0.01$. The initial data is a Gaussian distribution with $v_0 = 100$ and $\sigma = 20$ (for both the real and the imaginary parts of the field). A definite *power-law* fall-off is manifest at late-times. The power-law exponents are $-1.97, -3.94$ and -5.75 for the l=0,1 and 2 modes, respectively. These values are to be compared with the *analytically* predicted values of $-2.0, -4.0$ and -6.0.

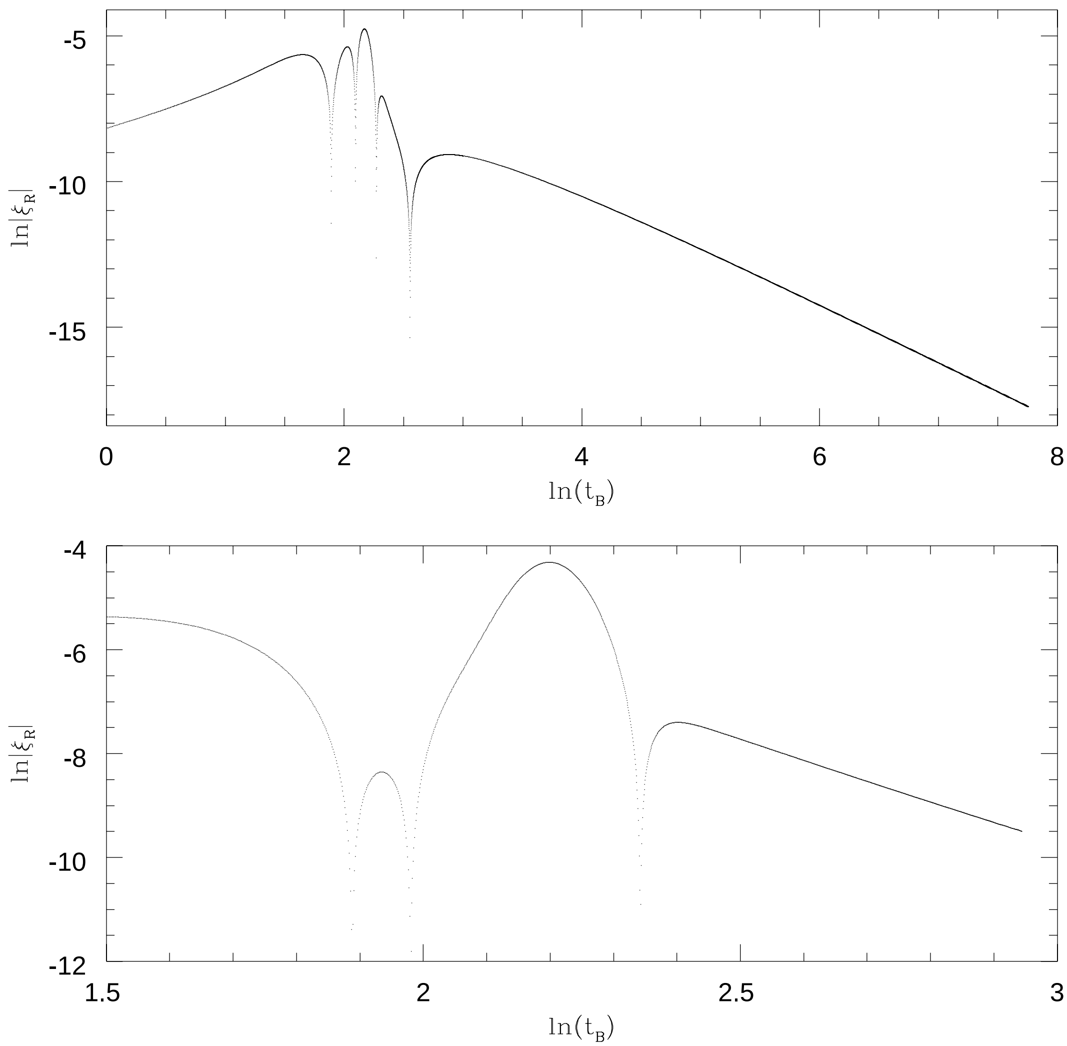

FIG. 4. Evolution of the charged test field $|\xi_R(r = 10, t)|$ on a *time-dependent* spacetime. The initial data for the background field ϕ are those of Fig. 2. (*non-collapsing* case). The initial data for the test field ξ is a Gaussian distribution (multiplied by an r^2 factor). A definite *power-law* fall-off is manifest at late-times. The power-law exponents are -1.99 for the l=0 mode (top panel) and -4.16 for the l=1 mode (bottom panel). These values are to be compared with the *analytically* predicted values of -2.0 and -4.0, respectively.

TABLES

TABLE I. Dependence of the dumping exponents at timelike infinity i_+ on eQ

e	Q_{BH}	$2\beta + 2$	Nonlinear exponents
0.50	-0.241	-1.97	-1.99
0.70	-0.344	-1.88	-1.93
0.85	-0.420	-1.70	-1.78
0.90	-0.443	-1.60	-1.65